Scattering of a Baseball by a Bat

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A ball can be hit faster if it is projected without spin but it can be hit farther if it is projected with backspin. Measurements are presented in this paper of the tradeoff between speed and spin for a baseball impacting a baseball bat. The results are inconsistent with a collision model in which the ball rolls off the bat and instead imply tangential compliance in the ball, the bat, or both. If the results are extrapolated to the higher speeds that are typical of the game of baseball, they suggest that a curveball can be hit with greater backspin than a fastball, but by an amount that is less than would be the case in the absence of tangential compliance.

I. INTRODUCTION

Particle scattering experiments have been conducted for many years to probe the structure of the atom, the atomic nucleus, and the nucleons. By comparison, very few scattering experiments have been conducted with macroscopic objects. In this paper we describe an experiment on the scattering of a baseball by a baseball bat, not to probe the structure of the bat but to determine how the speed and spin of the outgoing ball depends on the scattering angle. In principle the results could be used to determine an appropriate force law for the interaction, but we focus attention in this paper on directly observable parameters. The main purpose of the experiment was to determine the amount of backspin that can be imparted to a baseball by striking it at a point below the center of the ball. The results are of a preliminary nature in that they were obtained at lower ball speeds than those encountered in the field. As such, the experiment could easily be demonstrated in the classroom or repeated in an undergraduate laboratory as an introduction to scattering problems in general.

A golf ball is normally lofted with backspin so that the aerodynamic lift force will carry the ball as far as possible. For the same reason, a baseball will also travel farther if it is struck with backspin. It also travels farther if it is launched at a higher speed. In general there is a tradeoff between the spin and speed that can be imparted to a ball, which is affected in baseball by the spin and speed of the pitched ball. Sawicki, *et al.*[1], henceforth referred to as SHS, examined this problem and concluded that a curveball can be batted farther than a fastball despite the higher incoming and outgoing speed of the fastball. The explanation is that a curveball is incident with topspin and hence the ball is already spinning in the correct direction to exit with backspin. A fastball is incident with backspin so the spin direction needs to be reversed in order to exit with backspin. As a result, the magnitude of the backspin imparted to a curveball is larger than that imparted to a fastball for any given bat speed and impact point on the bat, even allowing for the lower incident speed of a curveball. According to SHS,[1] the larger backspin on a hit curveball more than compensates the smaller hit ball speed and travels farther than a fastball, a conclusion that has been challenged in the literature.[2]

SHS[1] assumed that a batted ball of radius r will roll off the bat with a spin ω given by $r\omega = v_x$ where v_x is the tangential speed of the ball as it exits the bat. However, a number of recent experiments[3,4,5,6,7] have shown that balls do not roll when they bounce. Rather, a ball incident obliquely on a surface will grip during the bounce and usually bounces with $r\omega > v_x$ if the angle of incidence is within about 45° to the normal. The actual spin depends on the tangential compliance or elasticity of the mating surfaces and is not easy to calculate accurately. For that reason we present in this paper measurements of speed, spin and rebound angle of a baseball impacting with a baseball bat. The implications for batted ball speed and spin are also described.

II. EXPERIMENTAL PROCEDURES

A baseball was dropped vertically onto a stationary, hand–held baseball bat to determine the rebound speed and spin as functions of (a) the scattering angle and (b) the magnitude and direction of spin of the incident ball. The impact distance from the longitudinal axis of the bat was varied on a random basis in order to observe scattering at angles up to about 120° away from the vertical. Measurements were made by filming each bounce with a video camera operating at 100 frames/s, although satisfactory results were also obtained at 25 frames/s. The bat chosen for the present experiment was a modified Louisville Slugger model R161 wood bat of length 84 cm (33 in.) with a barrel diameter of 6.67 cm (2 $\frac{5}{8}$ in.) and mass $M = 0.989$ kg (35 oz). The center of mass of the bat was located 26.5 cm from the barrel end of the bat. The moments of inertia about axes through the center of mass and perpendicular and parallel, respectively, to the longitudinal axis of the bat were 0.0460 and 4.39 x 10^{-4} kg-m^2. The ball was a Wilson A1010, having a mass 0.145 kg and diameter 7.2 cm.

The bat was held in a horizontal position by one hand and the ball was dropped from a height of about 0.8 m using the other hand. A plumb bob was used to establish a true vertical in the video image and to help align both the bat and the ball. The ball was dropped either with or without spin. In order to spin the ball, a strip of felt was wrapped around a circumference and the ball was allowed to fall vertically while holding the top end of the felt strip. A line drawn around a circumference was used to determine the ball orientation in each frame in order to measure its spin. The impact distance along the axis was determined by eye against marks on the barrel to within about 5 mm. If the ball landed 140 to 160 mm from the barrel end of the bat the bounce was accepted. Bounces outside this range were not analyzed.

The velocity of the ball immediately prior to and after impact was determined to within 2% by extrapolating data from at least three video frames before and after each impact. The horizontal velocity was obtained from linear fits to the horizontal coordinates of the ball and the vertical velocity was obtained from quadratic fits assuming a vertical acceleration of 9.8 m/s^2. Additional measurements were made by bouncing the ball on a hard wood floor to determine the normal and tangential COR, the latter defined below in Eq. 1, and a lower limit on the coefficient of sliding friction (μ_k) between the ball and the floor. The COR values were determined by dropping the ball with and without spin from a height of about 1.5 m to impact the floor at a speed of 5.6 ± 0.3 m/s. The incident ball spin was either zero, -72 ± 2 rad/s or $+68 \pm 3$ rad/s. The normal COR e_y was 0.59 ± 0.01, and the tangential COR e_x was 0.17 ± 0.03, corresponding to a rebound spin $\omega_2 \approx 0.16\omega_1$, where ω_1 is the incident spin. If a spinning baseball is dropped vertically onto a hard floor, then it would bounce with $\omega_2 = 0.29\omega_1$ if $e_x = 0$ (as assumed by SHS[1]). The lower limit on μ_k was determined by throwing the ball obliquely onto the floor at angles of incidence between 25° and 44° to the horizontal, at speeds from 3.5 to 4.2 m/s and with negligible spin. The value of μ_k was found from the data at low angles of incidence to be larger than 0.31 ± 0.02. At angles of incidence between 30° and 44° the ball did not slide throughout the bounce but gripped the floor during the bounce, with $e_x = 0.14 \pm 0.02$.

III. THEORETICAL BOUNCE MODELS

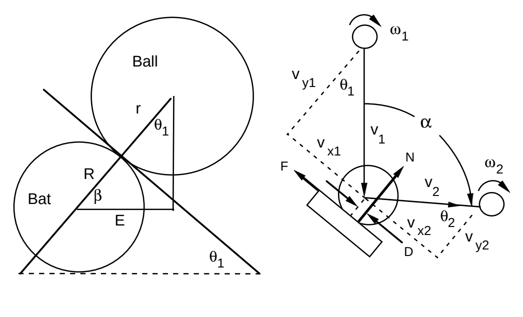

FIG. 1: Bounce geometry for a baseball of radius r, mass m falling vertically onto a bat of radius R, mass M with impact parameter E.

Consider the situation shown in Fig. 1 where a ball of radius r falls vertically onto a bat of radius

R. In a low speed collision the bat and the ball will remain approximately circular in cross section. If the impact parameter is E, then the line joining the bat and ball centers is inclined at an angle β to the horizontal where $\cos\beta = E/(r+R)$. The ball is incident at an angle $\theta_1 = 90-\beta$ to the line joining the bat and ball centers and will rebound at an angle θ_2. The ball is therefore scattered at an angle $\alpha = \theta_1+\theta_2$. During the collision, the ball experiences a tangential force F and a normal force N. For the low-speed collisions investigated here, the ball-bat force acts essentially at a point, so that the angular momentum of the ball about that point is conserved. Indeed, low-speed collisions of tennis balls are consistent with angular momentum conservation.[4] However, high-speed collisions of tennis balls are known not to conserve angular momentum.[5] A phenomenological way to account for non-conservation of angular momentum is to assume that the normal force N does not act through the center of mass of the ball but is displaced from it by the distance D,[4] as shown in Fig. 1 and discussed more fully below.

The collision is essentially equivalent to one between a ball and a plane surface inclined at angle θ_1 to the horizontal. Suppose that the ball is incident with angular velocity ω_1 and speed v_1. Let $v_{y1} = v_1\cos\theta_1$ denote the component of the incident ball speed normal to the surface and $v_{x1} = v_1\sin\theta_1$ denote the tangential component. The ball will bounce at speed v_{y2} in a direction normal to the surface, with tangential speed v_{x2} and angular velocity ω_2. If the bat is initially at rest, it will recoil with velocity components V_y and V_x respectively perpendicular and parallel to the surface, where the velocity components refer to the impact point on the bat. The recoil velocity at the handle end or the center of mass of the bat are different since the bat will tend to rotate about an axis near the end of the handle.

The bounce can be characterized in terms of three independent parameters: the normal coefficient of restitution (COR) $e_y = (v_{y2}-V_y)/v_{y1}$; the tangential COR, e_x, defined by

$$e_x = -\frac{(v_{x2} - r\omega_2 - [V_x - R\Omega])}{(v_{x1} - r\omega_1)} \tag{1}$$

where Ω is the angular velocity of the bat about the longitudinal axis immediately after the collision; and the parameter D. The two coefficients of restitution are defined respectively in terms of the normal and tangential speeds of the impact point on the ball, relative to the bat, immediately after and immediately before the bounce. The bounce can also be characterized in terms of apparent coefficients of restitution, ignoring recoil and rotation of the bat. That is, one can define the apparent normal COR $e_A = v_{y2}/v_{y1}$ and the apparent tangential COR, e_T, given by

$$e_T = -\frac{(v_{x2} - r\omega_2)}{(v_{x1} - r\omega_1)} \tag{2}$$

There are three advantages of defining apparent COR values in this manner. The first is that apparent COR values are easier to measure since there is no need to measure the bat speed and angular velocity before or after the collision (provided the bat speed is zero before the collision). The second advantage is that the batted ball speed can be calculated from the measured apparent COR values for any given initial bat speed simply by a change of reference frame. We show how this is done in Appendix B. The third advantage is that the algebraic solutions of the collision equations are considerably simplified and therefore more easily interpreted. Apparent and actual values of the COR are related by the expressions

$$e_A = \frac{(e_y - r_y)}{(1+r_y)} \tag{3}$$

and

$$e_T = \frac{(e_x - r_x)}{(1+r_x)} + \frac{5}{2}\left[\frac{D}{r}\right]\left(\frac{r_x}{1+r_x}\right)\frac{v_{y1}(1+e_A)}{(v_{x1} - r\omega_1)}, \tag{4}$$

where the recoil factors, r_y and r_x, are the ratios of effective ball to bat masses for normal and tangential collisions, respectively. An expression for r_y was derived by Cross:[8]

$$r_y = m\left(\frac{1}{M} + \frac{b^2}{I_0}\right) \tag{5}$$

and an expression for r_x is derived in Appendix A:

$$r_x = \frac{2}{7}m\left(\frac{1}{M} + \frac{b^2}{I_0} + \frac{R^2}{I_z}\right). \tag{6}$$

In these expressions, m is the ball mass, I_0 and I_z are the moments of inertia (MOI) about an axis through the center of mass and perpendicular and parallel, respectively, to the longitudinal axis of the bat, and

b is the distance parallel to the longitudinal axis between the impact point and the center of mass. For the bat used in the experiments at an impact distance 15 cm from the barrel end of the bat, r_y=0.188 and r_x=0.159, assuming the bat is free at both ends. The exit parameters of the ball are independent of whether the handle end is free or hand–held, as described previously by the authors.[9,10] Eq. 4 will not be used in this paper except for some comments in Sec. IV C and for comparison with SHS[1] who assumed in their calculations that $e_x = 0$ and $D = 0$, implying $e_T = -0.14$ for our bat. As discussed more fully in the next section, we find better agreement with our data when $e_T = 0$.

From the definition of the parameter D, the normal force exerts a torque resulting in a change in angular momentum of the ball about the contact point given by

$$(I\omega_2 + mrv_{x2}) - (I\omega_1 + mrv_{x1}) = -D \int N dt = -mD(1 + e_A)v_{y1} \tag{7}$$

where $I = \alpha mr^2$ is the moment of inertia of the ball about its center of mass. For a solid sphere, $\alpha = 2/5$, although Brody has recently shown that $\alpha \approx 0.378$ for a baseball.[11] Eqs. 2 and 7 can be solved to show that

$$\frac{v_{x2}}{v_{x1}} = \frac{(1 - \alpha e_T)}{(1 + \alpha)} + \frac{\alpha(1 + e_T)}{(1 + \alpha)} \left(\frac{r\omega_1}{v_{x1}}\right) - \frac{D(1 + e_A)}{r(1 + \alpha)} \left(\frac{v_{y1}}{v_{x1}}\right) \tag{8}$$

and

$$\frac{\omega_2}{\omega_1} = \frac{(\alpha - e_T)}{(1 + \alpha)} + \frac{(1 + e_T)}{(1 + \alpha)} \left(\frac{v_{x1}}{r\omega_1}\right) - \frac{D(1 + e_A)}{r(1 + \alpha)} \left(\frac{v_{y1}}{r\omega_1}\right) . \tag{9}$$

Eqs. 8 and 9, together with the definition of e_A give a complete description of the scattering process in the sense that, for given initial conditions, there are three observables, v_{y2}, v_{x2}, and ω_2, and three unknown parameters, e_A, e_T, and D, that can be inferred from a measurement of the observables. We have written Eq. 8 and 9 for the general case of nonzero D. However, as we will show in the next section, the present data are consistent with $D = 0$, implying conservation of the ball's angular momentum about the point of contact. The normal bounce speed of the ball is determined by e_A, while for $D \approx 0$ the spin and tangential bounce speed are determined by e_T and $r\omega_1/v_{x1}$. Depending on the magnitude and sign of the latter parameter, v_{x2} and ω_2 can each be positive, zero or negative. Eqs. 8 and 9 are generalizations of equations written down by Cross[4] for the special case of the ball impacting a massive surface and reduce to those equations when $e_A = e_y$ and $e_T = e_x$.

IV. EXPERIMENTAL RESULTS AND DISCUSSION

A. Determination of e_T

We initially analyze the data using Eqs. 8 and 9 assuming $D = 0$ and postpone for the time being a discussion of angular momentum conservation. Results obtained when the ball was incident on the bat without initial spin are shown in Fig. 2. The ball impacted the bat at speeds varying from 3.8 to 4.2 m/s but the results in Fig. 2 were scaled to an incident speed of 4.0 m/s by assuming that the rebound speed and spin are both directly proportional to the incident ball speed, as expected theoretically. An experimental value $e_A = 0.375 \pm 0.005$ was determined from results at low (back) scattering angles, and this value was used together with Eqs. 8 and 9 to calculate rebound speed, spin, and scattering angle as functions of the impact parameter for various assumed values of e_T. Best fits to the experimental data were found when $e_T = 0$ but reasonable fits could also be obtained with $e_T = 0 \pm 0.1$. Results obtained when the ball was incident with topspin or backspin are shown in Fig. 3. These results are not expected to scale with either the incident speed or incident spin and have not been normalized. Consequently the data show slightly more scatter than those presented in Figs. 2. The ball impacted the bat at speeds varying from 3.9 to 4.1 m/s and with topspin varying from 75 to 83 rad/s or with backspin varying from -72 to -78 rad/s. Simultaneous fits to all three data sets resulted in $e_A = 0.37 \pm 0.02$ and with $e_T = 0 \pm 0.02$. Using the recoil factors $r_y = 0.188$ and $r_x = 0.159$, our values for e_A and e_T imply $e_y = 0.63 \pm 0.01$ and $e_x = 0.16 \pm 0.02$. The result for e_x is consistent with that measured by impacting the ball onto a hard floor (0.17 ± 0.03) but the result for e_y is slightly higher, presumably because of the lower impact speed and the softer impact on the bat. On the other hand, Figs. 2 and 3 show that the measured ω_2 values are inconsistent with $e_T = -0.14$, which is the result expected for our bat if $e_x = 0$, as assumed by SHS.[1]

We next investigate the more general case in which angular momentum is not conserved by fitting the data to Eqs. 8 and 9 allowing both D and e_T as adjustable parameters. Fitting to all three data sets simultaneously, we find $e_T = 0 \pm 0.02$ and $D = 0.21 \pm 0.29$ mm. This justifies our earlier neglect of D and confirms that the data are consistent with angular momentum conservation. All calculations discussed below will assume $D = 0$.

It is possible to determine the incident and outgoing angles with respect to the normal, θ_1 and θ_2, from the measured quantities v_1, v_2, ω_1, ω_2, and α by applying angular momentum conservation about the contact point, Eq. 7, with D=0. Once θ_1 and θ_2 are known, it becomes possible to calculate other quantities of interest, such as the initial and final tangential

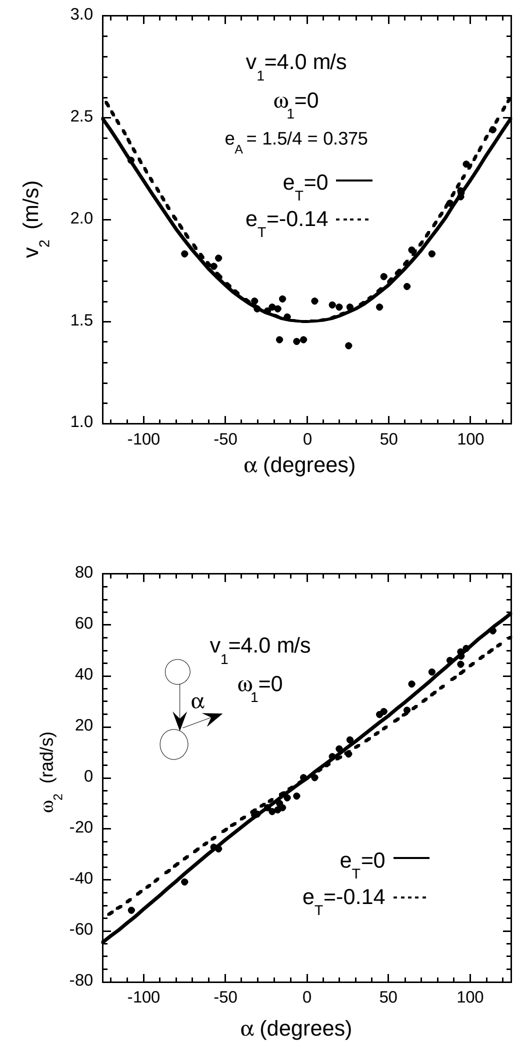

FIG. 2: Results with the ball incident with $\omega_1 = 0$, along with theoretical curves calculated with $e_T = 0$ or -0.14

velocities $v_x - r\omega$. The relationship between these quantities is shown in Fig. 4, where it is seen that the final velocities are clustered around zero, as would be expected for $e_T = 0$ (see Eq. 2). When plotted in this manner, it is quite clear that the data are completely inconsistent with $e_T = -0.14$. Note that the principal sensitivity to e_T comes from large values of $|v_{x1} - r\omega_1|$. This occurs whenever v_{x1} and ω_1 have the opposite sign, leading to a reversal of the spin and scattering angles that are negative for $\omega_1 > 0$

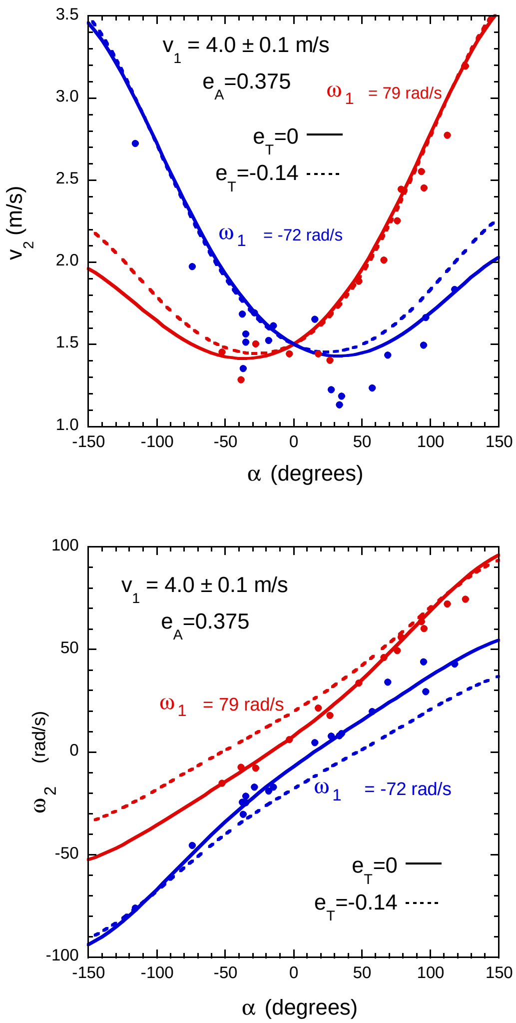

FIG. 3: Results with the ball incident with topspin or backspin, along with theoretical curves calculated with $e_T = 0$ or -0.14

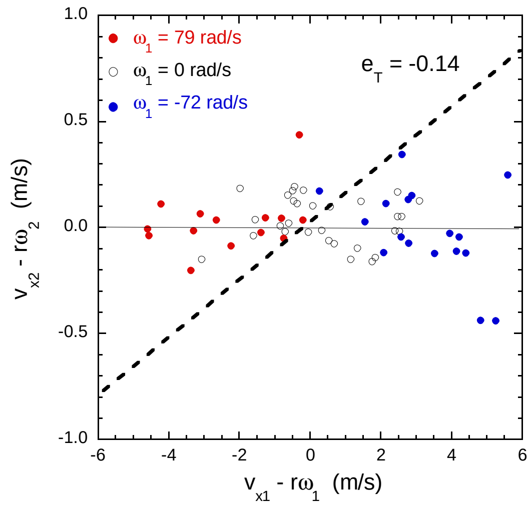

FIG. 4: Relationship between the final and initial tangential velocity of the ball. For $e_T = 0$, the final tangential velocity would be zero. The dashed line is the expected result for $e_T = -0.14$.

and positive for $\omega_1 < 0$. Indeed, Fig. 3 shows that those are exactly the regions of largest sensitivity to e_T.

B. Implications for batted balls

We next explore the implications of our results for the spin and speed of a batted ball, fully mindful that the present experiment was done at very low speeds compared to those appropriate for the game of baseball. The goal of this analysis is not to make any definite predictions about the spin and speed of a batted ball but to examine the consequences of a small positive value of e_x relative to the value $e_x = 0$ assumed by SHS.[1] Whether such a value of e_x is realized in a realistic ball-bat collision will have to await experiments at higher speed.

With that caveat, we first consider directly our own data in Fig. 5, where we plot ω_2 versus the impact parameter E, which is calculated from the inferred value of θ_1. These data demonstrate that for a given $E > 0$, a ball with initial topspin ($\omega_1 > 0$) has a larger outgoing backspin than a ball with initial backspin, in qualitative agreement with the argument of SHS.[1] To investigate the argument more quantitatively, we apply our formalism to the ball-bat scattering problem of SHS[1] to compare the final spin on a fastball to that on a curveball. In this problem, the bat and ball approach each other prior to the collision, thereby necessitating applying a change of reference frame to the formulas already developed. The relevant formulas, Eqs. B2–B4, are derived in Appendix B. We assume that the initial velocity of the bat is parallel to that of the ball, but displaced by the impact parameter E, as shown in

Fig. 6. The initial bat speed is 32 m/s (71.6 mph). The incident fastball has a speed of 42 m/s (94 mph) and spin of -200 rad/s (-1910 rpm), whereas the incident curveball has a speed of 35 m/s (78 mph) and a spin of +200 rad/s. Using values of the normal COR e_y assumed by SHS[1,12] the calculated final spin as a function of E is presented in Fig. 6 both for $e_T = 0$, as determined from the present measurements, and for $e_T = -0.14$, as assumed by SHS.[1]

Several important features emerge from this plot. First, the final spin ω_2 is *less sensitive* to the initial spin ω_1 for $e_T = 0$ than for $e_T = -0.14$. This result is consistent with Eq. 9, where the first term on the righthand side is larger for $e_T < 0$ than for $e_T = 0$. Our result means that the difference in backspin between a hit fastball and hit curveball is not as large as suggested by SHS.[1] Second, the gap between the spin on the curveball and fastball decreases as E increases, a feature that can be understood from the second term on the righthand side of Eq. 9. Since the initial speed is larger for the fastball than the curveball, the second term grows more rapidly for the fastball as E increases, and eventually the two curves cross at $E \approx 2.4$ in. Moreover, the rate at which the two curves converge is greater for $e_T = 0$ than for $e_T = -0.14$. Had the initial speeds been identical, the two curves would have been parallel. Third, for $E \gtrsim 0.5$ in. and independent of the sign of ω_1, ω_2 is *larger* when $e_T = 0$ than when $e_T < 0$, since ω_2 is mainly governed by the second term on the righthand side of Eq. 9. The increase in ω_2 is accompanied by a decrease in v_{x2}, as required by angular momentum conservation, and therefore by a slightly smaller scattering angle. However, the outgoing speed is dominated by the normal component, so the decrease in v_{x2} hardly affects the speed of the ball leaving the bat, at least for balls hit on a home run trajectory.

These results will have implications for the issue of whether an optimally hit curveball will travel farther than an optimally hit fastball. To investigate this in detail requires a calculation of the trajectory of a hit baseball, much as was done by SHS.[1] Such a calculation requires knowledge of the lift and drag forces on a spinning baseball. However, given the current controversy about these forces,[2] further speculation on this issue is beyond the scope of the present work.

It is interesting to speculate on the relative effectiveness of different bats regarding their ability to put backspin on a baseball. As we have emphasized, the effectiveness is determined by a single parameter, e_T, which in turn is related to the horizontal COR e_x and the recoil factor r_x. For a given e_x, a bat with a smaller r_x will be more effective than one with a larger r_x (see Eq. 4). Since r_x is dominated by the term involving R^2/I_z (see Eq. 6), one might expect it to be very different for wood and aluminum bats. Because of the hollow thin-walled construction of an aluminum bat, it will have a larger MOI about the longitudinal axis (I_z) than a wood bat of comparable mass and shape. This advantage is partially offset by the disadvantage of having a center of mass farther from the impact point (b is larger), which increases r_x. As a simple numerical exercise, we have investigated two bats having the shape of an R161, one a solid wood bat and the other a thin-walled aluminum bat. Both bats are 34 in. long and weigh 31.5 oz. The wood bat has I_0=2578 and I_z=18.0 oz-in^2 and the center of mass 22.7 in. from the knob. The aluminum bat has I_0=2985 and I_z=29.3 oz-in^2 and the center of mass 20.2 in. from the knob. With an impact 6 in. from the barrel end, where the bat diameter is 2.625 in, and assuming e_x=0.16, then e_T is -0.03 and 0, respectively, for the wood and aluminum bat. We conclude that, generally speaking, an aluminum bat is marginally more effective in putting backspin on the baseball than a wood bat of comparable mass and shape.

C. Insights into the scattering process

Besides the obvious practical implications of our result, it is interesting to ask what it teaches us about the scattering process itself. As mentioned earlier, our measured value $e_T = 0$ necessarily implies that $e_x \approx 0.16$. A value $e_x < 0$ would be obtained if the ball slides on the surface throughout the collision, whereas a value $e_x = 0$ would be obtained if the ball is rolling when it leaves the bat. However, a positive value of e_x necessarily implies tangential compliance in the ball, the bat, or both. A rigid baseball impacting on a rigid bat without any tangential compliance in the contact region will slide on the bat until the contact point comes to rest, in which case it will enter a rolling mode and it will continue to roll with zero tangential velocity as it bounces off the bat.[13] However, a real baseball and a real bat can store energy elastically as a result of deformation in directions both perpendicular

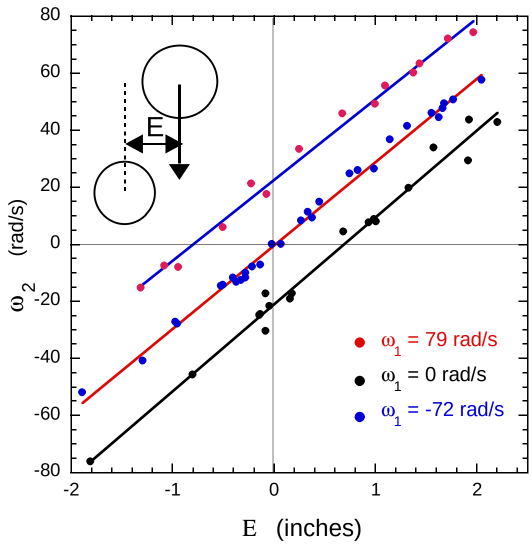

FIG. 5: Plot of the final spin ω_2 versus the impact parameter E, with $v_1 \approx 4.0$ m/s. These data clearly show that a ball with incident topspin ($\omega_1 > 0$) has larger final spin than a ball with incident backspin ($\omega_1 < 0$.) The lines are linear fits to the data.

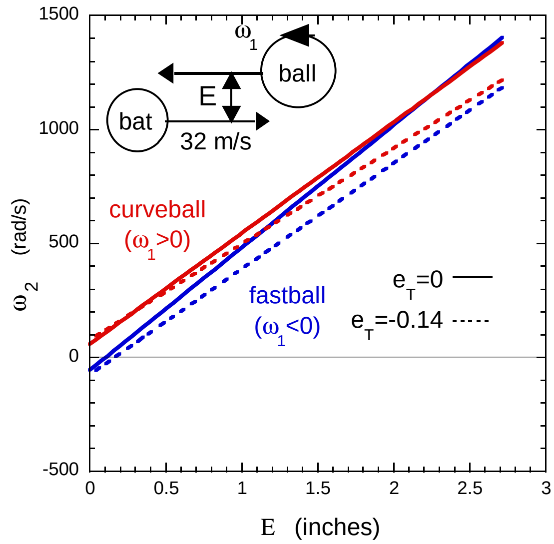

FIG. 6: Calculation of the outgoing spin on a fastball ($\omega_1 = -200$ rad/s, $v_1 = 42$ m/s) and a curveball ($\omega_1 = +200$ rad/s, $v_1 = 35$ m/s) for two different values of e_T.

and parallel to the impact surface. In that case, if tangential velocity is lost temporarily during the collision, then it can be regained from the elastic energy stored in the ball and the bat as a result of tangential deformation in the contact region. The ball will then bounce in an "overspinning" mode with $r\omega_2 > v_{x2}$ or with $e_x > 0$. The details of this process were first established by Maw, *et al.*[14,15] The effect is most easily observed in the bounce of a superball[16] which has tangential COR typically greater than 0.5.[3,4] A simple lumped-parameter model for the bounce of a ball with tangential compliance has been developed by Stronge.[17]

As mentioned above, the signature for continuous sliding throughout the collision is $e_x < 0$. Referring to Fig. 4, data with $e_x < 0$ would lie above or below the dashed line for values of $v_{x1} - r\omega_1$ greater than or less than zero, respectively. Not a single collision satisfies that condition in the present data set, suggesting that μ_k is large enough to bring the sliding to a halt. Therefore the scattering data themselves can be used to set a *lower limit* on μ_k, which must be at least as large as the ratio of tangential to normal impulse to the center of mass of the ball:

$$\mu_k \geq \frac{\int F dt}{\int N dt} = \frac{v_{x2} - v_{x1}}{(1 + e_A)v_{y1}} . \tag{10}$$

In Fig. 7 values of the RHS of Eq. 10 are plotted as a function of the initial ratio of tangential to normal speed. Using the results derived in Appendix A, it is straightforward to show that these quantities are linearly proportional with a slope equal to $(2/7)(1 + e_T)/(1 + e_A)$, provided the initial velocity ratio is below the critical value needed to halt the sliding. Stronge shows[17] and we confirm with our own formalism that the critical value is $(7/2)\mu_k(1 + e_A)(1 + r_x)$, which in our experiment assumes the numerical value $5.6\mu_k$. Above the critical value, the impulse ratio should be constant and equal to μ_k. Given that the data still follow a linear relationship up to an initial velocity ratio of 2.4, corresponding to an impulse ratio of 0.50, we conclude that $\mu_k \geq 0.50$. Indeed, if the actual μ_k were as small as 0.50, the critical value of the initial velocity ratio would be 2.8, which exceeds the maximum value in the experiment. The lower limit of 0.50 is larger than the lower limit of 0.31 that we measured from oblique collisions of a nonspinning ball with the floor. For that experiment, the angle with the horizontal needed to achieve continuous slipping is less than 20°, which is smaller than our minimum angle of 25°. Although no attempt was made to measure the ball-bat μ_k directly, our lower limit is consistent with $\mu_k = 0.50 \pm 0.04$ measured by SHS.[1]

Finally, we remark on our finding that the scattering data are consistent with $D \approx 0$, implying that the angular momentum of the ball is conserved about the initial contact point. At low enough initial speed, the deformation of the ball will be negligible, so that the ball and bat interact at a point and the angular momentum of the ball is necessarily conserved about that point. Evidentally, this condition is satisfied at 4 m/s initial speed. It is interesting to speculate whether this condition will continue to be satisfied at the much higher speeds relevant to the game of baseball, where the ball experiences considerable deformation and a significant contact area during the collision. Simple physics consideratons[4] suggest that it will not. A ball with topspin incident at an oblique angle will have a larger normal velocity at the leading edge than the trailing edge, resulting in a shift of the line of action of the normal force ahead of the center of mass of the ball ($D > 0$). A similar shift occurs when brakes are applied to a moving automobile, resulting in a larger normal force on the front wheels than on the back. Such a shift has been observed in high-speed collisions of tennis balls.[4,19] Whether a comparable shift occurs in high-speed baseball collisions will have to be answered with appropriate experimental data.

V. SUMMARY AND CONCLUSIONS

We have performed a series of experiments in which a baseball is scattered from a bat at an initial speed of about 4 m/s. For the particular bat that was used in the experiment, we find the horizontal apparent coefficient of restitution e_T is consistent with 0 and inconsistent with the value -0.14 expected if the ball is rolling at the end of its impact. These results necessarily imply tangential compliance in the ball, the bat, or both. We further find that the data are consistent with conservation of angular momentum of the ball about the contact point and with a coefficient of sliding friction between the ball and bat larger than 0.50. Our results suggest that a curveball can be hit with greater backspin than a fastball, but by an amount that is less than would be the case in the absence of tangential compliance. We note that since our investigations were done at low speed, one must proceed with caution before applying them to the higher speeds that are typical of the game of baseball.

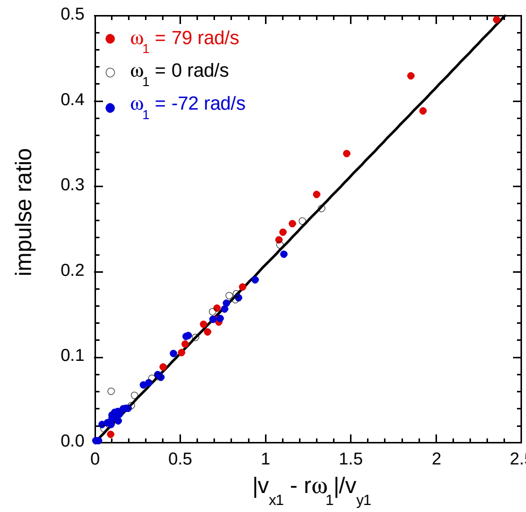

FIG. 7: The ratio of tangential to normal impulse, Eq.10, plotted as a function of the initial ratio of tangential to normal speed. The line is the expected impulse ratio for $e_A = 0.375$ and $e_T = 0$.

APPENDIX A: RELATIONSHIP BETWEEN e_T AND e_x

In this section, we derive for tangential collisions the relationship, Eqs. 4 and 6, between the apparent COR e_T and the COR e_x. The derivation follows closely that presented by Cross[8] for the relationship between e_A and e_y. We first solve the simple problem involving the collision of two point objects in one dimension. Object A of mass m and velocity v_1 is incident on stationary object B of mass M. Object A rebounds backwards with velocity v_2 and object B recoils with velocity V. Our sign convention is that v_1 is always positive and v_2 is positive if it is in the opposite direction to v_1. The collision is completely specified by the conservation of momentum

$$\int F dt = m(v_2 + v_1) = MV\,, \qquad \text{(A1)}$$

and the coefficient of restitution

$$e \equiv \frac{v_2 + V}{v_1},$$

where F is the magnitude of the force the two objects exert on each other. We define the apparent coefficient of restitution

$$e_A \equiv \frac{v_2}{v_1}, \tag{A2}$$

and we seek a relationship between e_A and e. Using Eq. A2, we write e as

$$e = e_A + \frac{V}{v_1},$$

then use Eq. A1 to find

$$\frac{V}{v_1} = (1 + v_2/v_1)\frac{m}{M},$$

from which we easily derive our desired expression

$$e_A = \frac{e - m/M}{1 + m/M}.$$

We next generalize this procedure for the collision of extended objects, as shown in Figs. 1 and 8. Specifically we consider the collision between a ball and a bat. A ball of mass m and radius r is incident obliquely in the xy-plane on a stationary bat of mass M and radius R at the impact point. Our coordinate system has its origin at the center of mass of the bat, with the z axis along the longitudinal axis and the x and y axes in the tangential and normal directions, respectively. The impact point P has the coordinates $(0, R, b)$. The ball is incident with angular velocity ω_1 and linear velocity components v_{x1} and v_{y1}; it rebounds with angular velocity ω_2 and linear velocity v_{x2} and v_{y2}, where the angular velocities are about the z axis. The bat recoils with CM velocity components V_x and V_y and with angular velocity about the CM with components Ω_x, Ω_y, and Ω_z. Let $\vec{v}_{p1}$, $\vec{v}_{p2}$, and $\vec{V}_p$ denote the pre- and post-impact velocities of the ball and the post-impact velocity of the bat at the point P, respectively. Since we are concerned with tangential collisions, we only consider the x components of these velocities, which are given by

$$\begin{aligned} v_{p1x} &= v_{x1} - r\omega_1 \\ v_{p2x} &= v_{x2} - r\omega_2 \\ V_{px} &= V_x + b\Omega_y + R\Omega_z \end{aligned} \tag{A3}$$

From the definitions in Eq. 1 and 2,

$$\begin{aligned} e_x &= -\frac{v_{p2x} - V_{px}}{v_{p1x}} \qquad \text{(A4)} \\ e_T &= -\frac{v_{p2x}}{v_{p1x}}. \end{aligned}$$

Applying the impulse-momentum expressions to the bat, we find

$$\begin{aligned} \int F dt &= MV_x \\ b\int F dt &= I_0\Omega_y \\ R\int F dt &= I_z\Omega_z, \end{aligned} \tag{A5}$$

where it has been assumed that the bat is symmetric about the z axis so that $I_x = I_y \equiv I_0$. Combining these equations with Eq. A3, we find

$$\int F dt = M_{ex}V_{px}, \tag{A6}$$

where M_{ex}, the bat effective mass in the x direction, is given by

$$\frac{1}{M_{ex}} = \frac{1}{M} + \frac{b^2}{I_0} + \frac{R^2}{I_z}. \tag{A7}$$

Applying the impulse-momentum expressions to the ball we find

$$\begin{aligned} \int F dt &= -m(v_{x2} - v_{x1}) \\ r\int F dt - D\int N dt &= \alpha m r^2(\omega_2 - \omega_1) \end{aligned} \tag{A8}$$

with $\alpha = 2/5$ for a uniform sphere. Noting that $\int N dt = (1 + e_A)mv_{y1}$ and combining the preceding equations with Eq. A3, we find

$$\int F dt = -m_{ex}(v_{p2x} - v_{p1x}) + \frac{m_{ex}Dv_{y1}(1 + e_A)}{r\alpha}, \tag{A9}$$

where m_{ex}, the ball effective mass in the x direction, is given by

$$m_{ex} = \frac{\alpha}{1 + \alpha}m. \tag{A10}$$

Combining the equations for the bat and ball, we arrive at

$$-m_{ex}(v_{p2x} - v_{p1x}) + \frac{m_{ex}Dv_{y1}(1 + e_A)}{r\alpha} = M_{ex}V_{px}, \tag{A11}$$

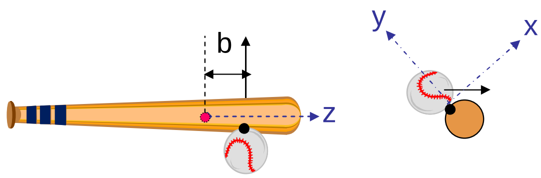

FIG. 8: Geometry for relating e_T and e_x. The origin of the coordinate system is at the center of mass of the bat, indicated by the red dot. The z axis points along the long axis towards the barrel. The x and y axes point, respectively, along the tangential and normal directions. The solid arrow indicates the initial velocity of the ball. The black dot labels the point of contact P between ball and bat. In the right-hand figure, the z axis points out of the plane.

which is analogous to the momentum conservation equation for point bodies, Eq. A1, provided the velocities refer those at the contact point P and the masses are effective masses. Following the derivation for point masses, we combined Eq. A11 with the definitions of e_x and e_T to arrive at the result

$$e_T \;=\; \frac{e_x - m_{ex}/M_{ex}}{1+m_{ex}/M_{ex}} + \left[\frac{D}{r\alpha}\right]\left(\frac{r_x}{1+r_x}\right)\frac{v_{y1}(1+e_A)}{(v_{x1}-r\omega_1)}\,. \tag{A12}$$

Defining $r_x = m_{ex}/M_{ex}$ and assuming $\alpha = 2/5$, then this equation, along with Eq. A7 and A10, is identical to Eqs. 4 and 6. Our results are equivalent to those used by SHS.[1] We note that Stronge[17] has derived an expression that is equivalent to Eq. 6 for the special case of a bat with zero length, implying $b = 0$, and $D = 0$. The present result is a generalization of Stronge's expression.

APPENDIX B: COLLISION FORMULAS IN THE LABORATORY REFERENCE FRAME

The formulas we have derived, Eqs. 8–9, for v_{x2} and ω_2 are valid in the reference frame in which the bat is initially at rest at the impact point. The usual (or "laboratory") frame that is relevant for baseball is the one where both the bat and ball initially approach each other. In this section, we derive formulas for v_{x2} and ω_2 in the laboratory frame. Our coordinate system is the same as that shown in Fig. 8, where the y axis is normal and the x axis is parallel to the ball-bat contact surface. In that system, the initial velocity components of the bat and ball at the impact point are denoted by (V_x, V_y) and $(v_{x1} - r\omega_1, v_{y1})$, respectively, where the usual situation has $V_y > 0$ and $v_{y1} < 0$. In the bat rest frame, the components of the ball initial velocity at the impact point are therefore $(v_{x1} - r\omega_1 - V_x, v_{y1} - V_y)$.

Applying the definitions of e_T and e_A, the components of the ball velocity after the collision are given by

$$v_{x2} - r\omega_2 - V_x \;=\; -e_T\,(v_{x1} - r\omega_1 - V_x)$$

and

$$v_{y2} - V_y \;=\; e_A\,(v_{y1} - V_y)\;,$$

which can be rearranged to arrive at

$$v_{x2} - r\omega_2 \;=\; -e_T\,(v_{x1} - r\omega_1) \;+\; (1+e_T)V_x \tag{B1}$$

and

$$v_{y2} \;=\; e_A v_{y1} \;+\; (1+e_A)V_y\,. \tag{B2}$$

Finally, we combine Eq. B1 with the expression for angular momentum conservation, Eq. 7, to find the following expressions for v_{x2} and w_2:

$$v_{x2} = v_{x1}\frac{(1-\alpha e_T)}{(1+\alpha)} + (r\omega_1 + V_x)\frac{\alpha(1+e_T)}{(1+\alpha)} \tag{B3}$$

and

$$r\omega_2 = r\omega_1\frac{(\alpha - e_T)}{(1+\alpha)} + (v_{x1} - V_x)\frac{(1+e_T)}{(1+\alpha)}\,. \tag{B4}$$

Eqs.B2–B4 are the results we seek. Eq. B2 has appeared in the literature many times.[18,19] To our knowledge, this is the first time anyone has written an explicit formula for v_{x2} and ω_2 in the laboratory frame. Although explicit formulas were not written down, the earlier works of SHS[1] and Watts and Baroni[20] are equivalent to ours for the special case $e_x = 0$, the latter being equivalent to $e_T = -0.137$ for our bat. Our formulas represent a generalization of their work to the case of arbitrary e_x.

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